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A two-phase free boundary problem for a nonlinear diffusion–convection equation

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Abstract

A two-phase free boundary problem associated with a diffusion–convection equation is considered. The problem is reduced to a system of nonlinear integral equations, which admits a unique solution for small times. The system admits an explicit two-component solution corresponding to a two-component shock wave of the Burgers equation. The stability of such a solution is also discussed.

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1. Introduction

Free boundary problems (FBP) have been the subject of several studies in the past due to their relevance in applications [1–6]. From the mathematical point of view FBP are initial/boundary value problems with a moving boundary [7]. The motion of the boundary is unknown (free boundary) and has to be determined together with the solution of the given partial differential equations. In recent studies [8–13] some free boundary problems for nonlinear evolution equations relevant in applications have been solved. In particular in [12] a one-phase FBP for the Rosen–Fokas–Yorstos equation [14, 15] was considered and shown to admit a unique solution for small times; moreover an exact travelling wave solution was obtained. On the other hand, the Rosen–Fokas–Yorstos model is a well-known nonlinear diffusion–convection equation modeling the flow of two immiscible fluids through a porous medium. It is therefore of interest to consider for this model a two-phase FBP, which is more complicated than its one-phase counterpart and requires a more elaborate theory.

In this paper we formulate and solve a two-phase free boundary problem characterized by the following system of nonlinear diffusion–convection equations,

$$\frac{\vartheta_{1t}}{\vartheta_1^2} = \mu_1 \vartheta_{1xx} - \vartheta_{1x}, \qquad \vartheta_1 \equiv \vartheta_1(x,t), \tag{1a}$$

defined over the domain $-\infty < x < s(t), t > 0$, and

$$\frac{\vartheta_{2t}}{\vartheta_2^2} = \mu_2 \vartheta_{2xx} - \vartheta_{2x}, \qquad \vartheta_2 \equiv \vartheta_2(x, t) \tag{1b}$$

over the domain $s(t) < x < +\infty, t > 0$, where

$$s(0) = b > 0.$$

Equations (1*a*) and (1*b*) have initial data given by

$$\vartheta_1(x, 0) = f_1(x) > 0, \qquad -\infty < x < b, \qquad f_1(b) = k \tag{2a}$$

$$\vartheta_2(x, 0) = f_2(x) > 0, \qquad b < x < +\infty, \qquad f_2(b) = k, \tag{2b}$$

with $f_2(x) \leqslant f_1(x)$, and the following set of boundary conditions:

$$\vartheta_1(-\infty, t) = \alpha_1, \qquad \vartheta_2(+\infty, t) = \beta_2, \tag{3a}$$

$$\vartheta_1(s(t), t) = \vartheta_2(s(t), t) = k, \tag{3b}$$

with $\alpha_1 > \beta_2 > 0$.

Equation (3*b*) in conjunction with a condition on the flow across the free boundary (see (3*c*) is sufficient to determine the motion of the free boundary $s(t)$. In principle, k in (3*b*) could be a given function of time. Here we will limit ourselves to the fundamental case k being a positive constant satisfying $\alpha_1 > k > \beta_2$.

In the above relations the dependent variables ϑ_j, $j = 1, 2$, are related to the volumetric fluids contents φ_j's through the relations [16]

$$\vartheta_j = \frac{1}{[1 + (\nu_j - 1)\varphi_j]}, \qquad j = 1, 2,$$

where ν_j, $j = 1, 2$, is the mobility of the fluid with respect to a reference ideal fluid.

Moreover, μ_j, $j = 1, 2$, is a parameter related to the relative strength of capillary to viscous forces.

The flow balance of the fluids across the free boundary can be written as

$$\eta_1 h_1(s(t), t) - \eta_2 h_2(s(t), t) = -(\eta_2\rho_2 - \eta_1\rho_1)\dot{s}(t) \tag{3c}$$

where

$$\eta_j = \frac{(\nu_j - 1)}{\nu_j}, \qquad j = 1, 2,$$

$$h_j = \frac{1}{\eta_j}(1 - \vartheta_j + \mu_j \vartheta_{jx}), \qquad j = 1, 2,$$

and ρ_j, $j = 1, 2$, is the density of the fluid, assumed to be constant.

Due to the boundary condition (3*b*), equation (3*c*) reduces to

$$-\lambda_1 \vartheta_{1x}(s(t), t) + \lambda_2 \vartheta_{2x}(s(t), t) = \dot{s}(t), \tag{4a}$$

where

$$\lambda_j = \frac{\mu_j}{a}, \qquad j = 1, 2, \qquad a = \eta_2\rho_2 - \eta_1\rho_1. \tag{4b}$$

Our analysis is based on the method developed in [12] for the solution of the one-phase free boundary problem.

In the next section, we reduce the above two-phase FBP to a system of coupled nonlinear integral equations. In section 3 we prove the existence and uniqueness of the solution for small intervals of time. In section 4 we finally construct an explicit solution of the two-phase free boundary problem and discuss its stability properties with respect to a small perturbation.

2. Linearization

We start our analysis by introducing the change of variables

$$\vartheta_j(x,t) = \psi_j(z,t), \qquad j = 1, 2, \qquad z \equiv z(x,t) \tag{5a}$$

with

$$z_{1x} = \frac{1}{\vartheta_1}, \qquad z_{1t} = \vartheta_1 - \mu_1\vartheta_{1x}, \qquad -\infty < x < s(t) \tag{5b}$$

$$z_{2x} = \frac{1}{\vartheta_2}, \qquad z_{2t} = \vartheta_2 - \mu_2\vartheta_{2x}, \qquad s(t) < x < +\infty, \tag{5c}$$

whose compatibility, $z_{xt} = z_{tx}$, is guaranteed by (1*a*) and (1*b*). The above transformation maps the system (1*a*)–(1*b*) into the following system of Burgers equations,

$$\begin{aligned} \psi_{1t} &= \mu_1\psi_{1zz} - 2\psi_1\psi_{1z}, \qquad -\infty < z < \bar{z}_1(t) \\ \psi_{2t} &= \mu_2\psi_{2zz} - 2\psi_2\psi_{2z}, \qquad \bar{z}_2(t) < z < +\infty \end{aligned} \tag{6}$$

where

$$\bar{z}_1(t) = \lim_{x\nearrow s(t)} z_1(x,t), \qquad \bar{z}_2(t) = \lim_{x\searrow s(t)} z_2(x,t). \tag{7}$$

It is worth noting that the two limits in (7) are different. In fact, when (5*a*)–(5*c*) are used together with the flux boundary condition (4*a*), we get

$$\dot{\bar{z}}_1(t) - \dot{\bar{z}}_2(t) = a\dot{s}(t), \tag{8}$$

which can be integrated giving

$$\bar{z}_1(t) - \bar{z}_2(t) = a[s(t) - b] \tag{9}$$

where we have put $\bar{z}_1(0) - \bar{z}_2(0) = 0$.

The above relations imply that equations (6) are defined over semi-infinite domains with distinct moving boundaries, given by (7). Relation (9) shows that the two moving boundaries, $\bar{z}_1(t)$ and $\bar{z}_2(t)$, are forced into a relative motion induced by the motion of the free boundary of the original problem. The relative velocity of the two boundaries is given by (9) and is proportional to the velocity $\dot{s}(t)$ of the free boundary motion.

The Burgers equations (6) are characterized by the set of initial data

$$\psi_j\left(z_{j_0}, 0\right) = f_j(x), \qquad j = 1, 2, \qquad z_{j_0} \equiv z_j(x, 0) \tag{10}$$

and by the boundary conditions

$$\psi_1(-\infty, t) = \alpha_1, \qquad \psi_2(+\infty, t) = \beta_2, \tag{11}$$

together with the boundary conditions at the moving boundaries

$$\psi_1(\bar{z}_1(t), t) = \psi_2(\bar{z}_2(t), t) = k \tag{12a}$$

and

$$-\lambda_1\psi_{1z}(\bar{z}_1(t), t) + \lambda_2\psi_{2z}(\bar{z}_2(t), t) = k\dot{s}(t), \tag{12b}$$

where (2*a*), (2*b*), (3*b*) and (4*a*) have been used.

The two-phase FBP for the nonlinear diffusion–convection equations (1*a*), (1*b*) has then been mapped into two distinct moving boundary problems for the Burgers equations (6) with initial data (10) and boundary conditions (11), (12*a*) and (12*b*).

In order to solve this problem we first observe that the Galilean transformation

$$\begin{cases} \psi_j - k \longrightarrow \psi_j, & j = 1, 2 \\ z + 2kt \longrightarrow z \end{cases}$$

leaves (6) invariant while implying trivial boundary condition in (12*a*).

Next, we introduce the generalized Hopf–Cole transformation [8]

$$\psi_j(z,t) = -\frac{\varphi_j(z,t)}{\left[C_j(t) + \frac{1}{\mu_j}\int_{\bar{z}_j(t)}^{z} \varphi_j(z',t)\,\mathrm{d}z'\right]}, \qquad j = 1, 2, \tag{13a}$$

$$\varphi_j(z,t) = C_j(t)\psi_j(z,t)\exp\left[-\frac{1}{\mu_j}\int_{\bar{z}_j(t)}^{z} \psi_j(z',t)\,\mathrm{d}z'\right], \qquad j = 1, 2, \tag{13b}$$

with

$$C_j(0) = 1. \tag{13c}$$

Under the above transformations we obtain from (6) the following system of linear heat equations,

$$\begin{aligned} \varphi_{1t} &= \mu_1 \varphi_{1zz}, \qquad -\infty < z < \bar{z}_1(t), \\ \varphi_{2t} &= \mu_2 \varphi_{2zz}, \qquad \bar{z}_2(t) < z < +\infty, \end{aligned} \tag{14}$$

together with the compatibility conditions

$$\dot{C}_j(t) = -\varphi_{jz}(\bar{z}_j(t), t), \qquad j = 1, 2. \tag{15}$$

Equations (14) are characterized by the initial data

$$\varphi_j(z_{j_0}, 0) = \psi_j(z_{j_0}, 0)\exp\left[-\frac{1}{\mu_j}\int_0^z \psi_j(z', 0)\,\mathrm{d}z'\right], \qquad j = 1, 2 \tag{16a}$$

and by the boundary conditions

$$\varphi_j(\bar{z}_j(t), t) = 0, \tag{16b}$$

$$\lambda_1 \frac{\varphi_{1z}(\bar{z}_1(t), t)}{C_1(t)} - \lambda_2 \frac{\varphi_{2z}(\bar{z}_2(t), t)}{C_2(t)} = k\dot{s}(t). \tag{16c}$$

The original FBP for the nonlinear diffusion–convection equations (1*a*), (1*b*) is then reduced to the solution of two moving boundary value problems for the linear heat equations (14), with initial data (16*a*) and boundary conditions at the moving boundary (16*b*) and (16*c*). These two problems are not independent of each other. They are connected via (16*c*) and (9), which puts a constraint on the relative motion of the boundaries.

We say that $\{\varphi_j(z,t), \bar{z}_j(t)\}_{j=1,2}$ form a solution of the above moving boundary problems for all $t < \sigma, 0 < \sigma < \infty$, when: (a) $\varphi_j(z,t)$, $(j = 1, 2)$ are solutions of (14) satisfying (16*a*)–(16*c*), they exist and are continuous together with their derivatives; (b) $\bar{z}_j(t)$ $(j = 1, 2)$ are continuously differentiable functions on $[0, \sigma)$ satisfying (9).

In order to prove the existence and uniqueness of the solution for $t < \sigma$, we assume that the initial data $f_j(x)$ $(j = 1, 2)$ given in (2*a*), (2*b*) are continuous with their derivatives; moreover they are bounded:

$$f_1(x) < \alpha_1, \qquad f_2(x) < \beta_2,$$

with α_1 and β_2 given by (3*a*).

Next we observe that the unknown functions $C_j(t)$ $(j = 1, 2)$ entering the transformation (13*a*)–(13*c*) satisfy the relation

$$-\lambda_1 \ln C_1(t) + \lambda_2 \ln C_2(t) = k[s(t) - b] \tag{17}$$

which is obtained from (16*c*) together with the compatibility condition (15). Integrating (15) and substituting back into (17) we obtain

$$-\lambda_1 \ln\left[1 - \int_0^t \varphi_{1z}(\bar z_1(\tau), \tau)\,\mathrm{d}\tau\right] + \lambda_2 \ln\left[1 - \int_0^t \varphi_{2z}(\bar z_2(\tau), \tau)\,\mathrm{d}\tau\right] = k[s(t) - b].$$

We now turn our attention to the solution of (14). To this end, we introduce the fundamental kernel of the heat equation

$$K_j(z - \xi, t - \tau) = \frac{1}{2\sqrt{\pi\mu_j}}\frac{1}{\sqrt{t - \tau}}\exp\left[-\frac{(z - \xi)^2}{4\mu_j(t - \tau)}\right], \qquad j = 1, 2$$

and integrate Green's identity for the heat equation

$$\frac{\partial}{\partial\xi}\left(K_j\frac{\partial\varphi_j}{\partial\xi} - \varphi_j\frac{\partial K_j}{\partial\xi}\right) - \frac{\partial}{\partial\tau}(K_j\varphi_j) = 0, \qquad j = 1, 2$$

over the domain $-\infty < \xi < \bar z_1(t)$ in the case $j = 1$ $[\bar z_2(t) < \xi < +\infty$ for $j = 2]$, $0 < \varepsilon < \tau < t - \varepsilon$ and let $\varepsilon \to 0$. Using (16*b*) and $K_j(z - \xi, 0) = \delta(z - \xi)$, we obtain

$$\varphi_1(z, t) = \int_{-\infty}^0 K_1(z - \xi, t)\varphi_1(\xi)\,\mathrm{d}\xi + \int_0^t K_1(z - \bar z_1(\tau), t - \tau)\varphi_{1z}(\bar z_1(\tau), \tau)\,\mathrm{d}\tau, \tag{18a}$$

$$\varphi_2(z, t) = \int_0^{+\infty} K_2(z - \xi, t)\varphi_2(\xi)\,\mathrm{d}\xi - \int_0^t K_2(z - \bar z_2(\tau), t - \tau)\varphi_{2z}(\bar z_2(\tau), \tau)\,\mathrm{d}\tau. \tag{18b}$$

We observe that on the right-hand side of (18*a*) [(18*b*)] the term $\varphi_{1z}(\bar z_1(t), t)$ $[\varphi_{2z}(\bar z_2(t), t)]$ is unknown. It is then convenient to take the z-derivative of both sides of (18*a*) [(18*b*)] and to take its limit as $z \nearrow \bar z_1(t)$ $[z \searrow \bar z_2(t)]$. By putting $w_j(t) = \varphi_{jz}(\bar z_j(t), t)$ $(j = 1, 2)$ we finally obtain (cf [13])

$$w_1(t) = 2\int_{-\infty}^0 K_1(\bar z_1(t) - \xi, t)\varphi_1'(\xi)\,\mathrm{d}\xi + 2\int_0^t K_{1z}(\bar z_1(t) - \bar z_1(\tau), t - \tau)w_1(\tau)\,\mathrm{d}\tau, \tag{19a}$$

$$w_2(t) = 2\int_{-\infty}^0 K_2(\bar z_2(t) - \xi, t)\varphi_2'(\xi)\,\mathrm{d}\xi + 2\int_0^t K_{2z}(\bar z_2(t) - \bar z_2(\tau), t - \tau)w_2(\tau)\,\mathrm{d}\tau, \tag{19b}$$

with

$$-\lambda_1 \ln\left[1 - \int_0^t w_1(\tau)\,\mathrm{d}\tau\right] + \lambda_2 \ln\left[1 - \int_0^t w_2(\tau)\,\mathrm{d}\tau\right] = k[s(t) - b]. \tag{19c}$$

Thus the solution of the moving boundary problems for the linear heat equations (14) has been reduced to the solution of the system of coupled nonlinear integral equations (19*a*)–(19*c*). Once the existence and uniqueness of the functions $w_j(t)$ $(j = 1, 2)$ are proved for $0 \leqslant t < \sigma$, the existence and uniqueness of $\varphi_j(z, t)$ $(j = 1, 2)$ then follow via (18*a*) and (18*b*). The solution of the two-phase FBP for the nonlinear diffusion–convection equations (1*a*) and (1*b*) then follows from (13*a*), (16*b*) and (5*a*); we point out that the unknown motion of the free boundary $s(t)$ is determined via (19*c*).

3. Contraction mapping

In order to analyze existence properties of $w_1(t)$ and $w_2(t)$ for $0 \leqslant t < \sigma$, we denote by S_M the closed sphere $\|w_j\| \leqslant M$ in the Banach space of functions $w_j(t)$ $(j = 1, 2)$ continuous for $0 \leqslant t < \sigma$, with the uniform norm $\|w_j\| = \text{l.u.b.}|w_j(t)|$. On the sphere S_M we define the mappings

$$y_j(t) = T w_j(t), \qquad j = 1, 2 \tag{20}$$

where $T w_1$ and $T w_2$ coincide with the right-hand side of (19*a*) and (19*b*) respectively. Let us now prove that T_j $(j = 1, 2)$ is a mapping of S_M into itself.

First, we go back to the original variables through the Galilean transformation

$$\begin{cases} \psi_j \longrightarrow \psi_j + k, & j = 1, 2 \\ z \longrightarrow z - 2kt \end{cases}$$

and then observe that (5*b*) and (5*c*), together with (7), imply

$$\bar{z}_j(t) = \int_0^t (\vartheta_j(s(\tau), \tau) - \mu_j \vartheta_{jx}(s(\tau), \tau))\, \mathrm{d}\tau + \int_0^b \frac{\mathrm{d}x'}{\vartheta_j(x', 0)}, \qquad j = 1, 2$$

which can be written as

$$\bar{z}_j(t) = A_j - \frac{\mu_j}{k} \int_0^t \frac{\varphi_{jz}(\bar{z}_j(\tau), \tau)}{C_j(\tau)}\, \mathrm{d}\tau, \qquad j = 1, 2$$

where (13*b*) has been used and $A_j = \int_0^b \frac{\mathrm{d}x'}{\vartheta_j(x', 0)}$.

We then can write

$$|\bar{z}_j(t)| \leqslant A_j + \frac{\mu_j}{k} B\sigma$$

together with

$$|\bar{z}_j(t) - \bar{z}_j(\tau)| \leqslant \frac{\mu_j}{k} B|t - \tau|, \tag{21}$$

where we put $B = \left\| \frac{w_j}{C_j} \right\|$.

We now consider the right-hand side of equation (20) in the case $j = 1$. It is shown in the appendix (cf (A.1)–(A.4) that

$$\|y_1\| = \|T w_1\| \leqslant 2A\, \mathrm{e}^{\frac{\|\psi_1\|}{\mu_1} A_1} + \frac{BM}{k\sqrt{\pi \mu_1}} \sqrt{\sigma} \tag{22}$$

where $A = \|\psi_1'\| + \frac{1}{\mu_1} \|\psi_1\|^2$.

We now define M as $M = \max(M_1, M_2)$, $M_j : M_j = 1 + 2A\, \mathrm{e}^{\frac{\|\psi_j\|}{\mu_j} A_j}$ $(j = 1, 2)$ and take $\sigma < \sigma_1$ where $\sigma_1 : BM\sqrt{\sigma_1} \leqslant k\sqrt{\pi \mu_1}$. It then follows from (22) that

$$\|y_1\| = \|T w_1\| \leqslant M. \tag{23a}$$

Along the same lines it is possible to show that

$$\|y_2\| = \|T w_2\| \leqslant M. \tag{23b}$$

Equations (23*a*) and (23*b*) imply that the mappings T_1 and T_2 are closed.

Next we prove that T_j $(j = 1, 2)$ is a contraction; i.e. given two solutions $w_j^{(1)}$ and $w_j^{(2)}$ of (20) with $\left\| w_j^{(1)} - w_j^{(2)} \right\| = \delta$, $\delta < 2M$, it follows that $\left\| T\left(w_j^{(1)} - w_j^{(2)}\right) \right\| \leqslant \vartheta\delta$ with $0 < \vartheta < 1$.

To this end, we denote by B_i appropriate positive constants and obtain the following relevant bounds (cf (A.5) and (A.6):

$$\left|\bar{z}_j^{(1)}(t) - \bar{z}_j^{(2)}(t)\right| \leqslant B_1 \delta\sigma t \qquad \left(B_1 = \frac{2\mu_j B}{k\gamma_j}\right) \tag{24a}$$

$$\left|\dot{\bar{z}}_j^{(1)}(t) - \dot{\bar{z}}_j^{(2)}(t)\right| \leqslant B_1 \delta\sigma. \tag{24b}$$

We now consider the case $j = 1$. From equations (20) and (19a) we can write

$$y_1^{(1)}(t) - y_1^{(2)}(t) = H_1 + H_2 \tag{25a}$$

where

$$H_1 = \frac{1}{\sqrt{\pi\mu_1 t}} \int_{-\infty}^{A_1} \varphi_1'(\xi) \left\{ \exp\left[-\frac{\left(\bar{z}_1^{(1)}(t) - \xi\right)^2}{4\mu_1 t}\right] - \exp\left[-\frac{\left(\bar{z}_1^{(2)}(t) - \xi\right)^2}{4\mu_1 t}\right]\right\} \mathrm{d}\xi, \tag{25b}$$

$$H_2 = -\frac{1}{\mu_1}\int_0^t w_1^{(1)}(\tau) \frac{\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau)\right)}{(t-\tau)} K_1\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau), t-\tau\right) \mathrm{d}\tau$$
$$+ \frac{1}{\mu_1}\int_0^t w_1^{(2)}(\tau) \frac{\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right)}{(t-\tau)} K_1\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau), t-\tau\right) \mathrm{d}\tau. \tag{25c}$$

In the appendix (cf (A.7) we derive

$$|H_1| \leqslant \frac{A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}A_1} B_1}{\sqrt{\pi\mu_1}} \delta\sigma^{3/2} \equiv B_2\delta\sigma^{3/2} \tag{26}$$

with $\sigma < \min(\sigma_1, \sigma_2)$, σ_2: $B_2\sigma_2^{3/2} < 1$.

The estimation of H_2 is obtained by writing

$$|H_2| \leqslant |V_1| + |V_2| + |V_3| \tag{27a}$$

where

$$V_1 = -\int_0^t \left(w_1^{(1)}(\tau) - w_1^{(2)}(\tau)\right)\left[\frac{\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau)\right)}{\mu_1(t-\tau)}\right]$$
$$\times K_1\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau), t-\tau\right) \mathrm{d}\tau, \tag{27b}$$

$$V_2 = -\int_0^t w_1^{(2)}(\tau)\left[\frac{\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau)\right)}{\mu_1(t-\tau)} - \frac{\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right)}{\mu_1(t-\tau)}\right]$$
$$\times K_1\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau), t-\tau\right) \mathrm{d}\tau, \tag{27c}$$

$$V_3 = -\int_0^t w_1^{(2)}(\tau)\left[\frac{\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right)}{\mu_1(t-\tau)}\right] K_1\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau), t-\tau\right)$$
$$\times \left\{1 - \exp\left[-\frac{\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right)^2 - \left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau)\right)^2}{4\mu_1(t-\tau)}\right]\right\} \mathrm{d}\tau. \tag{27d}$$

In the appendix the following bounds are obtained (cf (A.8)–(A.13)):

$$|V_1| \leqslant \frac{B}{k\sqrt{\pi\mu_1}}\delta\sqrt{\sigma} \equiv B_3\delta\sqrt{\sigma},$$
$$|V_2| \leqslant \frac{BM}{k\gamma_1\sqrt{\pi\mu_1}}\delta\sigma^{3/2} \equiv B_4\delta\sigma^{3/2}, \tag{28}$$
$$|V_3| \leqslant B_7\delta\sigma^{5/2},$$

where B_7 is defined in (A.13).

Combining (28), from (27*a*) we have

$$|H_2| \leqslant (B_3 + B_4 + B_7)\delta\sqrt{\sigma} \equiv B_8\delta\sqrt{\sigma}.$$

From the above relation, (25*a*) and (26), we finally get

$$\frac{\left\| y_1^{(1)} - y_1^{(2)} \right\|}{\delta} < (B_2 + B_8)\sqrt{\sigma} \equiv B_9\sqrt{\sigma}.$$

If we choose σ satisfying $\sigma < \min(\sigma_1, \sigma_2, \sigma_3)$, with

$$B_9\sqrt{\sigma_3} < 1,$$

it follows that T_1 is a contraction operator on S_M. Following the same lines it can be proven that T_2 is also a contraction operator on S_M. We also conclude that $y_1(t) = T_1 w_1(t)$ and $y_2(t) = T_2 w_2(t)$ exist and are the unique fixed points in S_M of T_1 and T_2 respectively, for $0 \leqslant t < \sigma$.

We have thus proven that the solution of the system of nonlinear integral equations (19*a*)–(19*c*) exists and is unique for a small interval of time, which in turn implies that the solution of the original two-phase FBP for equations (1*a*), (1*b*) exists and is unique for small times. In the next section we construct an explicit, particular solution of the two-phase FBP.

4. A two-component solution

In the following we show that there exists a particular solution $\{\vartheta_1(x,t), \vartheta_2(x,t), s(t)\}$ of the two-phase FBP for the system (1*a*) and (1*b*), corresponding to a two-component shock solution for the moving boundary problems associated with the Burgers equations (6). We write the usual shock solution of the first of the Burgers equations (6), compatible with (10) and (11) as

$$\psi_1(z,t) = \alpha_2 + \frac{(\alpha_1 - \alpha_2)}{\left[1 + \exp\frac{1}{\mu_1}(\alpha_1 - \alpha_2)(z - V_1 t - z_0')\right]} \tag{29a}$$

with

$$V_1 = \alpha_1 + \alpha_2, \qquad \alpha_1 > \alpha_2 > 0. \tag{29b}$$

The corresponding solution of the second of (6) satisfying (10) and (11) reads

$$\psi_2(z,t) = \beta_2 + \frac{(\beta_1 - \beta_2)}{\left[1 + \exp\frac{1}{\mu_2}(\beta_1 - \beta_2)(z - V_2 t - z_0'')\right]} \tag{30a}$$

with

$$V_2 = \beta_1 + \beta_2, \qquad \beta_1 > \beta_2 > 0. \tag{30b}$$

In the above relations α_2 and β_1 are constants to be determined.

We use (29*a*) [(30*a*)] on the interval $-\infty < z < \bar{z}_1(t)$ $[\bar{z}_2(t) < z < +\infty]$ and require

$$\psi_1(z,t) = 0, \quad z > \bar{z}_1(t) \qquad [\psi_2(z,t) = 0, z < \bar{z}_2(t)].$$

We now impose on $\psi_1(z,t)$ and $\psi_2(z,t)$ the condition at the free boundary (12*a*) and get

$$\bar{z}_1(t) - V_1 t = z_0' + \frac{\mu_1 s_0'}{(\alpha_1 - \alpha_2)}, \qquad \bar{z}_2(t) - V_2 t = z_0'' + \frac{\mu_2 s_0''}{(\beta_1 - \beta_2)},$$

which imply that the shock solutions are moving with the same velocities as the moving boundaries:

$$\dot{\bar{z}}_1(t) = V_1 \qquad \dot{\bar{z}}_2(t) = V_2. \tag{31}$$

Therefore, due to (8), their relative velocity is proportional to the velocity $\dot{s}(t)$ of the free boundary of the nonlinear diffusion–convection problem

$$V_1 - V_2 = a\dot{s}. \tag{32}$$

Next we observe that the boundary condition (12*b*) together with (29*a*), (30*a*), (32) and by using the boundary condition at the moving boundary (12*a*), gives the condition

$$\alpha_1\alpha_2 = \beta_1\beta_2. \tag{33}$$

Equations (31) and (33), when (29*b*) and (30*b*) are also used, determine the value of the constants α_2, V_1, V_2 in terms of β_1:

$$\alpha_2 = \frac{\beta_1\beta_2}{\alpha_1} \tag{34a}$$

$$V_1 = \alpha_1 + \frac{\beta_1\beta_2}{\alpha_1} \tag{34b}$$

$$V_2 = \beta_1 + \beta_2. \tag{34c}$$

Finally the solution of the two-phase FBP for the system (1*a*), (1*b*) is given in parametric form by

$$\vartheta_j(x,t) = \left(\frac{\partial z}{\partial x}\right)^{-1}, \qquad -\infty < z < \bar{z}_1(t) \quad \text{for } j = 1, \qquad \bar{z}_2(t) < z < \infty \quad \text{for } j = 2,$$

where, by virtue of (5*b*) and (5*c*), $z(x,t)$ solves

$$x = \int_0^z \psi_j(z',t)\,\mathrm{d}z', \qquad j = 1, 2$$

with $\psi_1(z,t)$ [$\psi_2(z,t)$] given by (29*a*) [(30*a*)].

The stability properties of the above two-component solution are discussed in the next section by studying the stability of the two-component shock wave solution of the Burgers equations (6).

5. Stability analysis and results

In order to study the stability of the particular solution $\{\psi_j(z,t), \bar{z}_j(t)\}_{j=1,2}$ analyzed in the previous section, we consider small perturbations affecting both the shocks and the motions of the boundaries. We set

$$\begin{aligned} \psi_j &= \hat{\psi}_j + \psi_j', \qquad j = 1, 2 \\ \bar{z}_j &= \hat{z}_j + z_j' \end{aligned} \tag{35}$$

where $\hat{\psi}_j$ $(j = 1, 2)$ is the shock solution satisfying $\hat{\psi}_j(\hat{z}_j(t), t) = 0$ and ψ_j', z_j' are small perturbations. It is important to observe that as a consequence of the perturbation on the boundaries $\bar{z}_j(t)$ $(j = 1, 2)$ we have

$$s = \hat{s} + s'. \tag{36}$$

By linearizing (6) around $\hat{\psi}_j$, we get

$$\Psi_{jt} = \mu_j\Psi_{jzz} - 2\hat{\psi}_j\Psi_{jz}, \qquad j = 1, 2 \tag{37}$$

where the position $\psi_j' = \Psi_{jz}$ has been made.

The boundary conditions (12*a*)–(12*b*), together with (35) and (36), give the conditions for $\Psi_j(z,t)$ at the free boundary:

$$-\frac{\lambda_1}{\alpha_1\alpha_2}\Psi_{1z}(\hat{z}_1(t),t)+\frac{\lambda_2}{\beta_1\beta_2}\Psi_{2z}(\hat{z}_2(t),t)=s'(t)$$

$$\begin{aligned}-\frac{\lambda_1}{\alpha_1\alpha_2}\frac{\partial}{\partial t}\Psi_{1z}\bigg|_{z=\hat{z}_1(t)}&+\frac{\lambda_2}{\beta_1\beta_2}\frac{\partial}{\partial t}\Psi_{2z}\bigg|_{z=\hat{z}_2(t)}\\ =-\frac{\lambda_1}{k}\left[\Psi_{1zz}+\frac{V_1}{\mu_1}\Psi_{1z}\right]_{z=\hat{z}_1(t)}&+\frac{\lambda_2}{k}\left[\Psi_{2zz}+\frac{V_2}{\mu_2}\Psi_{2z}\right]_{z=\hat{z}_2(t)}.\end{aligned} \tag{38}$$

The change of variables

$$\Psi_j(z,t)=\Psi_j(Z_j,t),\qquad Z_j=z-V_jt,\qquad j=1,2$$

maps (37) into

$$\Psi_{jt}=\mu_j\Psi_{j_{Z_jZ_j}}-(2\hat{\psi}_j-V_j)\Psi_{j_{Z_j}},\qquad j=1,2 \tag{39}$$

and (38) respectively into

$$-\frac{\lambda_1}{\alpha_1\alpha_2}\Psi_{1_{Z_1}}(0,t)+\frac{\lambda_2}{\beta_1\beta_2}\Psi_{2_{Z_2}}(0,t)=s'(t) \tag{40a}$$

and

$$\begin{aligned}-\frac{\lambda_1}{\alpha_1\alpha_2}\left[\Psi_{1Z_1t}-V_1\Psi_{1Z_1Z_1}\right]_{Z_1=0}&+\frac{\lambda_2}{\beta_1\beta_2}\left[\Psi_{2Z_2t}-V_2\Psi_{2Z_2Z_2}\right]_{Z_2=0}\\ =-\frac{\lambda_1}{k}\left[\Psi_{1Z_1Z_1}+\frac{V_1}{\mu_1}\Psi_{1Z_1}\right]_{Z_1=0}&+\frac{\lambda_2}{k}\left[\Psi_{2Z_2Z_2}+\frac{V_2}{\mu_2}\Psi_{2Z_2}\right]_{Z_2=0}.\end{aligned} \tag{40b}$$

We now solve (39) with the initial condition

$$\Psi_j(Z_j,0)=h_j(Z_j),\qquad j=1,2 \tag{40c}$$

and the asymptotically vanishing condition $\Psi_1\to 0$ $[\Psi_2\to 0]$ as $Z_1\to-\infty$ $[Z_2\to+\infty]$. In terms of the Laplace transform

$$\widetilde{\Psi}_j(Z_j,q)=\int_0^\infty \mathrm{e}^{-qt}\Psi_j(Z_j,t)\,\mathrm{d}t,\qquad j=1,2,$$

from (39) and (40*a*)–(40*c*), we get the solutions

$$\widetilde{\Psi}_1(Z_1,q)=\mathrm{e}^{-P_1(Z_1)}\left[c_1\,\mathrm{e}^{k_1Z_1}+\int_0^{Z_1}\frac{\mathrm{e}^{k_1(Z_1-\xi)}}{2k_1}H_1(\xi)\,\mathrm{d}\xi-\int_{-\infty}^{Z_1}\frac{\mathrm{e}^{-k_1(Z_1-\xi)}}{2k_1}H_1(\xi)\,\mathrm{d}\xi\right] \tag{41a}$$

and

$$\widetilde{\Psi}_2(Z_2,q)=\mathrm{e}^{-P_2(Z_2)}\left[c_2\,\mathrm{e}^{-k_2Z_2}-\int_0^{Z_2}\frac{\mathrm{e}^{-k_2(Z_2-\xi)}}{2k_2}H_2(\xi)\,\mathrm{d}\xi-\int_{Z_2}^{+\infty}\frac{\mathrm{e}^{k_2(Z_2-\xi)}}{2k_2}H_2(\xi)\,\mathrm{d}\xi\right] \tag{41b}$$

with

$$\begin{aligned}P_j(Z_j)&=\frac{1}{\mu_j}\int_0^{Z_j}\left(\frac{V_j}{2}-\hat{\psi}_j\right)\mathrm{d}Z_j\qquad j=1,2\\ k_1&=\left[\frac{(\alpha_1-\alpha_2)^2}{4\mu_1^2}+\frac{q}{\mu_1}\right]^{1/2}\end{aligned} \tag{41c}$$

$$k_2 = \left[\frac{(\beta_1-\beta_2)^2}{4\mu_2^2} + \frac{q}{\mu_2}\right]^{1/2}$$
$$H_j(Z_j) = -h_j(Z_j)\,e^{P_j(Z_j)}, \qquad j = 1, 2. \tag{41d}$$

c_1 and c_2 in (41a) and (41b) have to be determined via the boundary conditions (40a) and (40b).

The small perturbation $\psi_j'(z,t)$ ($j = 1, 2$) is finally obtained by inverting (41a) and (41b) and taking the z-derivative. When the large time behavior of $\psi_j'(z,t)$ is considered, we observe that all the contributions coming from the integral terms of (41a) and (41b) are asymptotically vanishing as $t \to +\infty$, since the branch points q_j of the solution are real and negative. We therefore conclude that the only possible source of asymptotically non–vanishing contributions to $\psi_j'(z,t)$ is determined by the positive singularities of c_1 and c_2.

When the boundary conditions (40a) and (40b) are imposed on (41a) and (41b), one obtains for c_1 and c_2 a system of the form

$$A_{11}c_1 + A_{12}c_2 = E_1 \qquad A_{21}c_1 + A_{22}c_2 = E_2 \tag{42}$$

where A_{ij} and E_i ($i, j = 1, 2$) depend on the variable q, and on the parameters $k_j, \lambda_j, \mu_j, \alpha_j$ and β_j. Their explicit forms are

$$A_{11} = \frac{\lambda_1 V_1}{2\mu_1\alpha_1\alpha_2} - \frac{\lambda_1 k_1}{\alpha_1\alpha_2},$$

$$A_{12} = \frac{\lambda_2 V_2}{2\mu_1\beta_1\beta_2} + \frac{\lambda_2 k_2}{\beta_1\beta_2},$$

$$A_{21} = \frac{\mu_1}{\alpha_1\alpha_2}\left[-\frac{V_1}{2\mu_1}q + k_1 q + \frac{\alpha_1\alpha_2 V_1}{\mu_1^2} - \frac{V_1^2 k_1}{\mu_1} + \frac{V_1^3}{4\mu_1^2} + V_1 k_1^2\right] + \frac{\mu_1}{k}\left[-\frac{V_1^2}{4\mu_1^2} + \frac{\alpha_1\alpha_2}{\mu_1^2} + k_1^2\right],$$

$$A_{22} = -\frac{\mu_2}{\beta_1\beta_2}\left[-\frac{V_2}{2\mu_2}q + k_2 q + \frac{\beta_1\beta_2 V_2}{\mu_2^2} + \frac{V_2^2 k_2}{\mu_2} + \frac{V_2^3}{4\mu_2^2} + V_2 k_2^2\right] - \frac{\mu_2}{k}\left[-\frac{V_2^2}{4\mu_2^2} + \frac{\beta_1\beta_2}{\mu_2^2} + k_2^2\right],$$

with k_1 and k_2 given by (41c) and (41d); moreover it is

$$E_1 = \frac{\lambda_1 I_1}{\alpha_1\alpha_2}\left[\frac{V_1}{4k_1\mu_1} + \frac{1}{2}\right] - \frac{\lambda_2 I_2}{\beta_1\beta_2}\left[\frac{V_2}{4k_2\mu_2} - \frac{1}{2}\right] + \tilde{S}(q),$$

$$\begin{aligned} E_2 = {} & \frac{\mu_1 V_1}{\alpha_1\alpha_2}h_1(0) - \frac{\mu_1 I_1}{\alpha_1\alpha_2}\left[\frac{V_1 q}{4k_1\mu_1} + \frac{q}{2} - \frac{\alpha_1\alpha_2 V_1}{2k_1\mu_1^2} - \frac{V_1^2}{2\mu_1} - \frac{V_1^3}{8k_1\mu_1^2} - \frac{V_1}{2}\right] \\ & - \frac{\mu_2 V_2}{\beta_1\beta_2}h_2(0) + \frac{\mu_2 I_2}{\beta_1\beta_2}\left[\frac{V_2 q}{4k_2\mu_2} + \frac{q}{2} - \frac{\beta_1\beta_2 V_2}{2k_2\mu_2^2} + \frac{V_2^2}{2\mu_2} - \frac{V_2^3}{8k_2\mu_2^2} - \frac{V_2}{2}\right] \\ & + \frac{\mu_1}{k}h_1(0) - \frac{\mu_1 I_1}{k}\left[\frac{V_1^2}{8k_1\mu_1^2} - \frac{\alpha_1\alpha_2}{2k_1\mu_1^2} - \frac{1}{2}\right] \\ & - \frac{\mu_2}{k}h_2(0) - \frac{\mu_2 I_2}{k}\left[\frac{V_2^2}{8k_2\mu_2^2} - \frac{\beta_1\beta_2}{2k_2\mu_2^2} - \frac{1}{2}\right], \end{aligned}$$

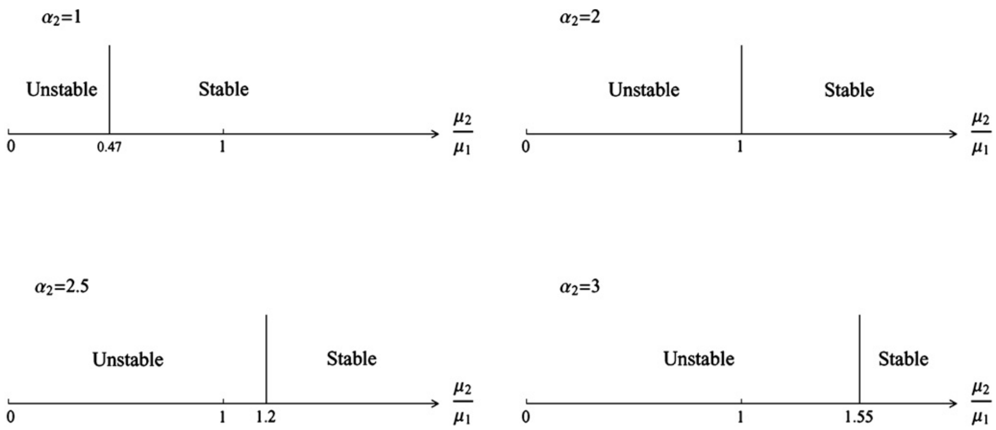

Figure 1. Regions of stability and instability for different values of the parameter α_2 and for $k = 3, \mu_1 = 0.6, \beta_2 = 5, \lambda_1 = 1$.

where $\tilde{S}(q)$ is the Laplace transform of $s'(t)$ and

$$I_1 = \int_{-\infty}^{0} \mathrm{e}^{k_1\xi} H_1(\xi)\,\mathrm{d}\xi \qquad I_2 = \int_{0}^{\infty} \mathrm{e}^{-k_2\xi} H_2(\xi)\,\mathrm{d}\xi.$$

The determinant Δ of the system (42) ($\Delta = A_{11}A_{22} - A_{21}A_{12}$) can be evaluated as a function of q for different values of the parameters $\lambda_j, \mu_j, \alpha_j, \beta_j$.

The zeros of Δ (for positive q) determine the instability of the shock solution $\hat{\psi}_j$ $(j = 1, 2)$ with respect to the small perturbation ψ_j'.

The results of numerical computations indicate that the behavior of Δ as a function of q is strongly influenced by the values of the parameters $\lambda_j, \mu_j, \alpha_j$ and β_j $(j = 1, 2)$. These results are shown in figure 1 where the parameter α_2 identifies the regions of stability and instability of the shock wave in terms of the ratio $\frac{\mu_2}{\mu_1}$:

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Appendix

In the following we derive some relevant inequalities of section 2. We consider the norm $\|y_1\|$ and prove relation (22). From (16*a*) we observe that

$$\begin{aligned} |\varphi_1'(z)| &= \left|\psi_1'(z) - \frac{1}{\mu_1}\psi_1^2(z)\right| \exp\left(-\frac{1}{\mu_1}\int_{A_1}^{z}\psi_1(z')\,\mathrm{d}z'\right) \\ &\leqslant \left(\|\psi_1'\| + \frac{1}{\mu_1}\|\psi_1\|^2\right)\exp\left[-\frac{\|\psi_1\|}{\mu_1}(z - A_1)\right] \\ &\equiv A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}(A_1 - z)}. \end{aligned} \qquad \text{(A.1)}$$

Moreover, we note that the two terms on the right-hand side of (19*a*) satisfy

$$2\left|\int_{-\infty}^{A_1} K_1(\bar{z}_1(t)-\xi,t)\varphi_1'(\xi)\,\mathrm{d}\xi\right|$$
$$\leqslant \frac{A}{\sqrt{\pi\mu_1}}\frac{1}{\sqrt{t}}\int_{-\infty}^{A_1}\exp\left[-\frac{(\bar{z}_1(t)-\xi)^2}{4\mu_1 t}\right]\exp\left[\frac{\|\psi_1\|}{\mu_1}(A_1-\xi)\right]\mathrm{d}\xi$$
$$\leqslant 2A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}A_1}, \tag{A.2}$$

$$2\left|\int_0^t K_{1z}(\bar{z}_1(t)-\bar{z}_1(\tau),t-\tau)w_1(\tau)\,\mathrm{d}\tau\right|$$
$$\leqslant \frac{1}{2\sqrt{\pi}\mu_1^{3/2}}\int_0^t \frac{|\bar{z}_1(t)-\bar{z}_1(\tau)|}{|t-\tau|^{3/2}}\exp\left[-\frac{(\bar{z}_1(t)-\bar{z}_1(\tau))^2}{4\mu_1(t-\tau)}\right]|w_1(\tau)|\mathrm{d}\tau$$
$$\leqslant \frac{BM}{k\sqrt{\pi\mu_1}}\sqrt{\sigma}. \tag{A.3}$$

When the above relations are used we get

$$\|y_1\| \leqslant 2A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}A_1} + \frac{BM}{k\sqrt{\pi\mu_1}}\sqrt{\sigma}. \tag{A.4}$$

In order to prove (24*a*) and (24*b*) we call

$$R_1(t) = \left(\frac{w_1^{(1)}(t)}{C_1^{(1)}(t)} - \frac{w_1^{(2)}(t)}{C_1^{(2)}(t)}\right) \tag{A.5}$$

and observe that

$$\frac{\|R_1\|}{\left\|w_1^{(1)}-w_1^{(2)}\right\|} \leqslant \frac{2B}{\gamma_1}\sigma \tag{A.6}$$

where $\gamma_1 = \left\|C_1^{(1)} - C_1^{(2)}\right\|$.

Our final task is to derive the bounds on $|H_1|$ and $|H_2|$. We start our analysis with (25*b*) and write

$$|H_1| \leqslant \frac{A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}A_1}}{\sqrt{\pi\mu_1 t}}\int_{A_1-\xi_0^{(1)}}^{A_1-\xi_0^{(2)}}\exp\left(-\frac{y^2}{4\mu_1 t}\right)\mathrm{d}y$$

where $\xi_0^{(j)} = \bar{z}_1^{(j)}(t) - 2\|\psi_1\|t$, $(j = 1, 2)$, and (A.1) has also been used. The above relation together with (24*a*) implies

$$|H_1| \leqslant \frac{A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}A_1}}{\sqrt{\pi\mu_1 t}}\left|\bar{z}_1^{(1)}(t)-\bar{z}_1^{(2)}(t)\right|$$
$$\leqslant \frac{A\,\mathrm{e}^{\frac{\|\psi_1\|}{\mu_1}A_1}B_1}{\sqrt{\pi\mu_1}}\delta\sigma^{3/2} \equiv B_2\delta\sigma^{3/2}. \tag{A.7}$$

In order to study $|H_2|$ we consider first (27*b*) and get

$$|V_1| \leqslant \frac{\delta}{2\sqrt{\pi}\mu_1^{3/2}}\int_0^t\left|\frac{\bar{z}_1^{(1)}(t)-\bar{z}_1^{(1)}(\tau)}{t-\tau}\right|\frac{\mathrm{d}\tau}{\sqrt{t-\tau}}$$
$$< \frac{B}{k\sqrt{\pi\mu_1}}\delta\sqrt{\sigma} \equiv B_3\delta\sqrt{\sigma}. \tag{A.8}$$

Next, from (27*c*) we can write

$$|V_2| \leqslant \frac{M}{2\sqrt{\pi}\mu_1^{3/2}} \int_0^t \left| \frac{\left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(2)}(t)\right) - \left(\bar{z}_1^{(1)}(\tau) - \bar{z}_1^{(2)}(\tau)\right)}{(t-\tau)} \right| \frac{\mathrm{d}\tau}{\sqrt{t-\tau}}$$
$$\leqslant \frac{M}{2\sqrt{\pi}\mu_1^{3/2}} \int_0^t \left|\dot{\bar{z}}_1^{(1)}(\alpha) - \dot{\bar{z}}_1^{(2)}(\alpha)\right| \frac{\mathrm{d}\tau}{\sqrt{t-\tau}}$$
$$< \frac{BM}{k\gamma_1\sqrt{\pi\mu_1}}\delta\sigma^{3/2} \equiv B_4\delta\sigma^{3/2} \tag{A.9}$$

where the mean-value theorem, (A.5), (A.6) and (24*b*) have been used.

Finally for the estimate of V_3 we put in (27*d*)

$$Q = \frac{\left[\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right)^2 - \left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau)\right)^2\right]}{4\mu_1(t-\tau)}$$
$$= \frac{1}{4\mu_1(t-\tau)}\left[\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(1)}(t)\right) - \left(\bar{z}_1^{(2)}(\tau) - \bar{z}_1^{(1)}(\tau)\right)\right]$$
$$\times \left[\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right) + \left(\bar{z}_1^{(1)}(t) - \bar{z}_1^{(1)}(\tau)\right)\right]. \tag{A.10}$$

By using (24*a*) and (21) we get

$$|Q| \leqslant \frac{BB_1\delta\sigma t}{k} < \frac{BB_1}{k}\delta\sigma^2 \equiv B_5\delta\sigma^2. \tag{A.11}$$

On the other hand, from (21) it also follows that

$$|Q| \leqslant \frac{\mu_1 B^2}{2k^2}|t-\tau| < \frac{\mu_1 B^2}{2k^2} \equiv B_6. \tag{A.12}$$

From (27*d*) we then get

$$|V_3| \leqslant \frac{M}{2\sqrt{\pi}\mu_1^{3/2}} \int_0^t \left| \frac{\left(\bar{z}_1^{(2)}(t) - \bar{z}_1^{(2)}(\tau)\right)}{(t-\tau)} \right| \frac{|1-\mathrm{e}^{-Q}|}{\sqrt{t-\tau}}\,\mathrm{d}\tau$$
$$\leqslant \frac{MB}{k\sqrt{\pi\mu_1}}|Q|\,\mathrm{e}^{|Q|}\sqrt{\sigma}$$
$$\leqslant \frac{MBB_5\,\mathrm{e}^{B_6}}{k\sqrt{\pi\mu_1}}\delta\sigma^{5/2} \equiv B_7\delta\sigma^{5/2} \tag{A.13}$$

where the inequality $|1-\mathrm{e}^{-Q}| \leqslant |Q|\,\mathrm{e}^{|Q|}$ has been used.

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